A GENERALIZATION OF THE FOURIER COSINE SERIES*

BY

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It is well known that on the interval $0 \leqq x \leqq \pi$ large classes of functions arbitrary except for certain restrictions as to oscillation or as to continuity and the existence and continuity of derivatives or as to some similar property can be developed into series in terms of the functions $\{\cos nx\}$. The series are of the type

$$(1) \qquad f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots,$$

where any coefficient a_n may be obtained by the formal process of multiplying equation (1) through by $\cos nx\,dx$ and integrating term by term:

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx\,dx.$$

The present paper considers the problem of developing arbitrary functions throughout the same interval in terms of the second set of functions $\{\cos \lambda_n x\}$, where λ_n is very near to n. Under suitable restrictions it is proved that the two developments have essentially the same convergence properties.† We

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shall say that two series

$$\sum_{i=1}^{\infty} a_i \, \bar{u}_i, \qquad \sum_{i=j}^{\infty} b_i \, u_i$$

have essentially the same convergence properties when and only when the series

$$\sum_{i=1}^{\infty} (a_i \, \bar{u}_i - b_i \, u_i)$$

converges absolutely and uniformly to the sum zero.

The set of functions

$$\frac{1}{\pi}, \ \frac{2}{\pi} \cos x, \ \frac{2}{\pi} \cos 2x, \ \cdots$$

is a normal and orthogonal set on the interval $0 \leqq x \leqq \pi$. Any set of functions $\{\bar{u}_n\}$ continuous on a finite interval is said to be normal and orthogonal on that interval if and only if

$$\int \bar{u}_i \, \bar{u}_j \, dx = \delta_{ij} \qquad\qquad (i, j = 1, 2, 3, \cdots);$$

here and throughout this paper δ_{ij} is the Kronecker symbol which is unity or zero according as i and j are or are not equal. In this integral and below we omit the argument, which is x in every case, and also omit the limits of integration, which are the ends of the interval considered. We shall prove that corresponding to the set of functions

$$\frac{1}{\pi} \cos \lambda_0 \, x, \ \frac{2}{\pi} \cos \lambda_1 \, x, \ \frac{2}{\pi} \cos \lambda_2 \, x, \ \cdots$$

there is another set of functions such that the two sets are biorthogonal for $0 \leqq x \leqq \pi$. Two sets of continuous functions $\{u_n\}$ and $\{v_n\}$ are biorthogonal on an interval if and only if

$$\int u_i \, v_j \, dx = \delta_{ij} \qquad\qquad (i, j = 1, 2, \cdots).$$

We shall prove a general theorem concerning the generalization of expansions in terms of any uniformly bounded normal orthogonal set $\{\bar{u}_n\}$, by considering a set $\{u_n\}$ where u_n is a function *neighboring* to $\bar{u}_n$. Briefly described, the method is to expand the u_n into series in terms of the $\bar{u}_n$, to invert the corresponding system of equations and expand the $\bar{u}_n$ in terms of the u_n, and finally to substitute the latter series into the expansion of the arbitrary function in terms of the $\bar{u}_n$. This gives the expansion of $f(x)$ in terms of the u_n. The form of presentation of the material is slightly changed by the introduction of a set of functions $\{v_n\}$ such that $\{u_n\}$ and $\{v_n\}$ are biorthogonal sets. The various steps in the proof will seem much less artificial if the general method is kept in mind.

After the general theorem is proved, application is made to the cosine series. We add that the method seems to be of very wide applicability; it is hoped later to give further applications to the Sturm-Liouville and other series developments. Throughout the present paper there is more regard for simplicity than for ultimate generality.

I. A GENERAL THEOREM

THEOREM I: *Suppose that $\{\bar{u}_n\}$ is a set of uniformly bounded normal orthogonal functions in an interval, and that in this interval $\{u_n\}$ is a set of uniformly bounded continuous functions each of which can be developed into a series*

$$(2) \qquad u_n = \sum_{k=1}^{\infty} (c_{nk} + \delta_{nk}) \bar{u}_k \qquad (n = 1, 2, \cdots),$$

where the coefficients have the values

$$(3) \qquad c_{nk} + \delta_{nk} = \int u_n \bar{u}_k \, dx.$$

Suppose further that the three series

$$(4) \qquad \sum_{i,k=1}^{\infty} c_{ik}^2, \quad \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} c_{ik}^2 \right)^{\frac{1}{2}}, \quad \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} c_{ik}^2 \right)^{\frac{1}{2}}$$

converge and that the value of the first is less than unity.

Then there exists a set of functions $\{v_i\}$ such that $\{u_i\}$ and $\{v_i\}$ are biorthogonal sets:

$$(5) \qquad \int u_i v_j \, dx = \delta_{ij} \qquad (i, j = 1, 2, \cdots).$$

Furthermore, if $f(x)$ is any function integrable and with an integrable square (in the sense of Lebesgue), then the two series

$$(6) \qquad f(x) \sim \sum_{i=1}^{\infty} a_i \bar{u}_i,$$

$$(7) \qquad f(x) \sim \sum_{i=1}^{\infty} b_i u_i,$$

where

$$(8) \qquad \begin{aligned} a_i &= \int f \bar{u}_i \, dx, \\[2mm] b_i &= \int f v_i \, dx, \end{aligned}$$

have essentially the same convergence properties.

The sign $\sim$ is used simply to indicate that the coefficients a_i and b_i are given

by (8). We shall give later some of the more immediate consequences of the fact that (6) and (7) have essentially the same convergence properties.

To prepare for the inversion of system (2), which is an important step in the proof of Theorem I, we shall use the following

LEMMA: *If for the system*

$$
\begin{aligned}
(1 + c_{11})x_1 + \quad c_{12}\ x_2 + \quad c_{13}\ x_3 + \cdots &= c_1, \\
c_{21}\ x_1 + (1 + c_{22})x_2 + \quad c_{23}\ x_3 + \cdots &= c_2, \\
c_{31}\ x_1 + \quad c_{32}\ x_2 + (1 + c_{33})x_3 + \cdots &= c_3, \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots &,
\end{aligned}
$$

(9)

the series

$$
\gamma^2 = \sum_{j=1}^{\infty} c_j^2, \qquad p^2 = \sum_{i,j=1}^{\infty} c_{ij}^2
$$

converge, and if $p^2 < 1$, then the system has one solution $\{x_i\}$ and only one solution for which

$$
\sum_{i=1}^{\infty} x_i^2
$$

converges.[*]

The lemma is proved by the method of successive approximations, setting

$$
\begin{aligned}
x_i^{(1)} &= c_i, \\
x_i^{(2)} &= c_i - [c_{i1}\ x_1^{(1)} + c_{i2}\ x_2^{(1)} + \cdots], \\
\cdots &\cdots \cdots \cdots \cdots \cdots \cdots \cdots, \\
x_i^{(n+1)} &= c_i - [c_{i1}\ x_1^{(n)} + c_{i2}\ x_2^{(n)} + \cdots], \\
\cdots &\cdots \cdots \cdots \cdots \cdots \cdots \cdots.
\end{aligned}
$$

From the well-known inequality

(10)
$$
\sum_{k=1}^{n} |\alpha_k\ \beta_k| \leqq \left(\sum_{k=1}^{n} \alpha_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \beta_k^2\right)^{\frac{1}{2}}
$$

(which holds for all sets of numbers $\{\alpha_k\}$ and $\{\beta_k\}$) and from the convergence of the series whose sums are denoted by γ^2 and p^2, it follows that the series

$$
x_i = x_i^{(1)} + (x_i^{(2)} - x_i^{(1)}) + (x_i^{(3)} - x_i^{(2)}) + \cdots \qquad (i = 1, 2, \cdots)
$$

are all absolutely convergent and that the system $\{x_i\}$ as thus defined is a

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[*] This lemma is included implicitly in some results of Hilb, S i t z u n g s b e r i c h t e d e r P h y s i k a l i s c h - m e d i z i n i s c h e n S o z i e t ä t i n E r l a n g e n, vol. 40 (1908), pp. 1–6. Hilb makes use of the theory of bilinear forms in infinitely many variables, so that his proof differs in form (but not in substance) from the proof here presented. We indicate the proof because of its simplicity and because we shall use later the inequalities obtained in the course of the proof.

solution of (9). If we introduce the notation $p_i^2 = \sum_{j=1}^{\infty} c_{ij}^2$, it is readily proved by further use of the same inequality that

(11)
$$|x_i - c_i| \leqq \frac{\gamma p_i}{1 - p},$$

$$\sum_{i=1}^{\infty} |x_i - c_i|^2 \leqq \frac{\gamma^2 p^2}{(1 - p)^2}.$$

The series $\sum_{i=1}^{\infty} x_i^2$ converges, for we obtain the relations*

$$|x_i| \leqq |x_i - c_i| + |c_i|,$$

$$\sum_{i=1}^{\infty} x_i^2 \leqq \sum_{i=1}^{\infty} (|x_i - c_i| + |c_i|)^2 \leqq 2 \sum_{i=1}^{\infty} (x_i - c_i)^2 + 2 \sum_{i=1}^{\infty} c_i^2.$$

There is no other solution of system (9) such that $\sum_{i=1}^{\infty} x_i^2$ converges. For if there were, we should have a solution $\{y_i\}$ of the homogeneous system corresponding to (9) and such that the series

$$Y^2 = \sum_{i=1}^{\infty} y_i^2$$

converges. In the kth equation of the homogeneous system, if we transpose y_k to the right-hand member, we obtain

$$y_k^2 \leqq Y^2 p_k^2,$$

and hence we also have

$$\sum_{k=1}^{\infty} y_k^2 \leqq Y^2 p^2.$$

In this same manner we have

$$\sum_{k=1}^{\infty} y_k^2 \leqq Y^2 p^{2n},$$

where n is any positive integer, and hence $y_k = 0$ ($k = 1, 2, \cdots$).

We apply this lemma (the proof of which is now complete) to determine the set of functions $\{v_k\}$ of Theorem I. Let us assume a development of the type

(12) $v_k = \sum_{i=1}^{\infty} (d_{ki} + \delta_{ki}) \bar{u}_i,$ where $d_{ki} + \delta_{ki} = \int v_k \bar{u}_i \, dx.$

The condition that the v_k have the desired property (5) is given formally from the developments (2) and (12) by the conditions

(13) $c_{ki} + d_{ik} + c_{k1} d_{i1} + c_{k2} d_{i2} + c_{k3} d_{i3} + \cdots = 0$

for every i and k. When i is kept fixed, this is precisely system (9); the

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* By use of the inequality

$$(\alpha + \beta)^2 \leqq 2 (\alpha^2 + \beta^2).$$

conditions of the lemma are fulfilled by virtue of the hypothesis of Theorem I, and therefore there exists a solution $(d_{i1}, d_{i2}, d_{i3}, \cdots)$,* with the properties stated in the lemma. If we choose that solution and use inequalities (11) and the convergence of series (4), we actually arrive at a system of continuous functions $\{r_k\}$ which satisfy (5) and are given by (12) and (13).

Our preparation for the inversion of system (2) is nearly complete. We introduce the notation

(14) $$\epsilon_{ik} = c_{ki} + d_{ik} + c_{1i} d_{1k} + c_{2i} d_{2k} + c_{3i} d_{3k} + \cdots$$

and shall prove $\epsilon_{ik} = 0$ for every i and k. In the set of equations

$$\epsilon_{i1} = c_{1i} + d_{i1} + c_{1i} d_{11} + c_{2i} d_{21} + c_{3i} d_{31} + \cdots,$$

$$\epsilon_{i2} = c_{2i} + d_{i2} + c_{1i} d_{12} + c_{2i} d_{22} + c_{3i} d_{32} + \cdots,$$

$$\epsilon_{i3} = c_{3i} + d_{i3} + c_{1i} d_{13} + c_{2i} d_{23} + c_{3i} d_{33} + \cdots,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

multiply the ith equation by c_{ni} and sum by columns, making use of (13) and (14). We obtain the result

$$c_{k1} \epsilon_{i1} + c_{k2} \epsilon_{i2} + c_{k3} \epsilon_{i3} + \cdots = - \epsilon_{ik}.$$

The formal work is readily justified by the convergence of series (4). Moreover, the series $\sum_{k=1}^{\infty} \epsilon_{ik}^2$ converges, from the definition (14) and from (11) applied to the set $(d_{i1}, d_{i2}, d_{i3}, \cdots)$ as a solution of (13), so the set $(\epsilon_{i1}, \epsilon_{i2}, \epsilon_{i3}, \cdots)$ forms a solution of the homogeneous system corresponding to (9) and hence $\epsilon_{ik} = 0$ for every i and k.

We are now in a position to invert system (2). We write the equations

$$c_{k1} + d_{1k} + c_{11} d_{1k} + c_{21} d_{2k} + c_{31} d_{3k} + \cdots = 0,$$

$$c_{k2} + d_{2k} + c_{12} d_{1k} + c_{22} d_{2k} + c_{32} d_{3k} + \cdots = 0,$$

$$c_{k3} + d_{3k} + c_{13} d_{1k} + c_{23} d_{2k} + c_{33} d_{3k} + \cdots = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

multiply the ith equation by $\bar{u}_i$ and sum by columns. Justification of the formal work is immediate, and we obtain the desired developments

$$\bar{u}_k - u_k = \sum_{i=1}^{\infty} d_{ik} u_i \qquad (k = 1, 2, 3, \cdots).$$

Let us suppose, now, $f(x)$ to be any function integrable and with an integrable square in the sense of Lebesgue. We find the coefficients (8) by multi-

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* For a non-vanishing finite determinant, the numbers $d_{ik} + \delta_{ik}$ are the quotients of the cofactors of the $c_{ik} + \delta_{ik}$ by the value of the determinant.

plying (12) through by $f(x)$ and integrating term by term. This yields the result

(15) $$b_k - a_k = d_{k1} a_1 + d_{k2} a_2 + d_{k3} a_3 + \cdots .^*$$

The difference of the series (6) and (7) may be written

(16)
$$\sum_{i=1}^{\infty} (a_i \bar{u}_i - b_i u_i) = \sum_{i=1}^{\infty} a_i (\bar{u}_i - u_i) + \sum_{i=1}^{\infty} (a_i - b_i) u_i$$

$$= -\sum_{i=1}^{\infty} a_i \left(\sum_{j=1}^{\infty} c_{ij} \bar{u}_j \right) - \sum_{i=1}^{\infty} (d_{i1} a_1 + d_{i2} a_2 + \cdots) \left(\bar{u}_i + \sum_{j=1}^{\infty} c_{ij} \bar{u}_j \right),$$

which series converges absolutely and uniformly and has the sum zero. We make use of the convergence of the series $\sum_{i=1}^{\infty} a_i^2$, which may be proved easily from the relation

(17) $$\int (f - a_1 \bar{u}_1 - a_2 \bar{u}_2 - \cdots - a_n \bar{u}_n)^2 \, dx \geqq 0 .$$

Hence, *series* (6) *and* (7) *have the same convergence properties, in the sense that* (16) *converges absolutely and uniformly to zero.* It follows immediately that properties of absolute convergence, convergence, summability, and divergence obtain for one series as for the other, and likewise the properties of uniform convergence in the entire interval considered or in any sub-interval, uniform summability, and also of term-by-term integrability. Whenever the two series (6) and (7) are convergent, summable, or properly divergent, their sums are the same. The nature of the approximating functions and of their approach to the limit (in case of convergence) at a point of continuity or of discontinuity of $f(x)$ is essentially the same for (6) as for (7). In particular if Gibbs's phenomenon occurs for (6) it also occurs for (7). The reader will notice various other properties common to the sets $\{\bar{u}_i\}$ and $\{u_i\}$, such as the existence or non-existence of a continuous function for which the formal series does not converge at every point.

The conditions of Theorem I concerned with the series (4) can be made

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* If we use the convergence of $\Sigma_{i=1}^{\infty} a_i^2$, which is proved from (17); if we write the equations

$$b_1 - a_1 = d_{11} a_1 + d_{12} a_2 + d_{13} a_3 + \cdots ,$$
$$b_2 - a_2 = d_{21} a_1 + d_{22} a_2 + d_{23} a_3 + \cdots ,$$
$$b_3 - a_3 = d_{31} a_1 + d_{32} a_2 + d_{33} a_3 + \cdots ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot ,$$

multiply the ith equation by c_{ik}, and sum by columns, we obtain

$$a_k - b_k = c_{1k} b_1 + c_{2k} b_2 + c_{3k} b_3 + \cdots .$$

Hence if the $\{b_i\}$ are all zero, so are all the $\{a_i\}$. The converse is evident from (15).

We also add that if the series $\Sigma_{i=1}^{\infty} |a_i|$ converges, so does the series $\Sigma_{i=1}^{\infty} |b_i|$, and conversely.

more restrictive so that the convergence properties of (6) and (7) will be essentially the same for all functions absolutely integrable in the sense of Lebesque, or on the other hand can be made less restrictive and still yield a result say for functions such that the series of absolute values of the coefficients $\sum_{i=1}^{\infty}|a_i|$ converges.

We add the remark that the relation between the $\{u_i\}$ and the $\{v_i\}$ is essentially reciprocal—we may obtain corresponding results for expansions in terms of the $\{v_i\}$. Theorem I is readily extended, moreover, by taking as point of departure developments in terms of functions $\{\bar{u}_i\}$ which are not supposed orthogonal.

II. Application to the Fourier cosine series

For the interval $0 \leqq x \leqq \pi$, we have mentioned the normal orthogonal system of uniformly bounded continuous functions

$$(18) \qquad \frac{1}{\pi}, \frac{2}{\pi}\cos x, \frac{2}{\pi}\cos 2x, \frac{2}{\pi}\cos 3x, \cdots.$$

Application of Theorem I will yield the following theorem.

Theorem II. *The expansion of any function $f(x)$ integrable and with an integrable square in terms of the functions*

$$\frac{1}{\pi}\cos\lambda_0 x, \frac{2}{\pi}\cos\lambda_1 x, \frac{2}{\pi}\cos\lambda_2 x, \frac{2}{\pi}\cos\lambda_3 x, \cdots$$

on the interval $0 \leqq x \leqq \pi$, where

$$(19) \qquad \lambda_0^2 + 4(\lambda_1 - 1)^2 + 4(\lambda_2 - 2)^2 + 4(\lambda_3 - 3)^2 + \cdots < \frac{1}{\pi}$$

and where

$$(20) \qquad \sum_{n=1}^{\infty} n^2 |\lambda_n - n|$$

converges, has essentially the same convergence properties as the expansion of $f(x)$ in terms of the functions (18).

It is of course true that our set of functions $\{u_m\}$ can be developed in terms of the functions $\{\bar{u}_m\}$, so it merely remains to prove the convergence of the series (4) and that the value of the first of these series is less than unity.

Inequality (17) gives us the relation

$$\sum_{k=1}^{\infty} c_{mk}^2 \leqq \int (\bar{u}_m - u_m)^2 \, dx \qquad (m = 1, 2, 3, \cdots),$$

and in the present case if $m \neq 1$ and if we replace m by $n + 1$, this integral

becomes

$$(21) \quad \frac{4}{\pi^2} \int_0^\pi (\cos \lambda_n x - \cos nx)^2 \, dx$$

$$= \frac{4}{\pi^2} \int_0^\pi [4 \sin^2 \tfrac{1}{2}(\lambda_n + n)x \cdot \sin^2 \tfrac{1}{2}(\lambda_n - n)x] \, dx.$$

But we always have the relation $|\sin \alpha| \leq |\alpha|$, and hence we obtain

$$\sum_{k=1}^\infty c_{n+1,\,k}^2 \leq 4\pi(\lambda_n - n)^2 \qquad (n = 1, 2, 3, \cdots)$$

If we omit the factor 4 from the right-hand member we obtain the proper formula for $n = 0$. Satisfaction of (19) therefore means that we have satisfied the following requirement of Theorem I

$$\sum_{m,\,k=1}^\infty c_{mk}^2 < 1.$$

It remains to be shown that the last two series in (4) converge. From the formula

$$(22) \qquad c_{n+1,\,k+1} = \frac{4}{\pi^2} \int_0^\pi (\cos \lambda_n x - \cos nx) \cos kx \, dx$$

$$(n = 1, 2, \cdots; \ k = 1, 2, \cdots)$$

integration by parts twice yields the formula

$$(23) \quad \frac{\pi^2}{4} c_{n+1,\,k+1} = -\left[(\lambda_n \sin \lambda_n x - n \sin nx) \frac{\cos kx}{k^2} \right]_0^\pi$$

$$+ \int_0^\pi (\lambda_n^2 \cos \lambda_n x - n^2 \cos nx) \frac{\cos kx}{k^2} \, dx.$$

To deal with this integral, we write

$$(24) \quad \lambda_n^2 \cos \lambda_n x - n^2 \cos nx = (\lambda_n^2 - n^2) \cos \lambda_n x + n^2(\cos \lambda_n x - \cos nx).$$

If $n > 2$, we find with the aid of (19) the obvious relation

$$|\lambda_n + n| < 2n + 2 < n^2,$$

whence

$$|\lambda_n^2 - n^2| < n^2 |\lambda_n - n|;$$

we use this inequality in considering the first term of the right-hand member of (24). The last term of the right-hand member of (24) can be transformed as in (21), and its absolute value is therefore not greater than $n^2 \pi |\lambda_n - n|$.

The first term in the right-hand member of (23) with the factor $\cos k\pi/k^2$ omitted, reduces to $\lambda_n (\sin \lambda_n \pi - \sin n\pi)$. When $n > 1$, we have $|\lambda_n| \leq n^2$, and therefore

$$|\lambda_n (\sin \lambda_n \pi - \sin n\pi)| \leq n^2 \pi |\lambda_n - n|.$$

From these inequalities for the absolute value of the right-hand member of (23) and from the corresponding results for $k = 0$ and $n \leqq 2$ we prove the convergence of

$$\sum_{n,\,k=1}^{\infty} |c_{nk}|$$

by means of the convergence of (20) and hence* prove the convergence of the last two series of (4) and complete the proof of Theorem II.

We add the remark that a closer evaluation of c_{nk}, and also the consideration of functions $u_n = \mu_n \cos \lambda_n x$ where μ_n is properly determined, will give broader restrictions than (19) and (20).

HARVARD UNIVERSITY,
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* Here we make use of the inequality

$$\sum_{i=1}^{n} |\alpha_i| = \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 \right]^{\frac{1}{2}} \geqq \left[\sum_{i=1}^{n} \alpha_i^2 \right]^{\frac{1}{2}}.$$